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Branes as stable holomorphic line bundles on the non-commutative torus

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Abstract: It was suggested by A. Kapustin that turning on a B-field, and allowing some discrepancy between the left and and right-moving complex structures, must induce an identification of B-branes with holomorphic line bundles on a non-commutative complex torus. The stability condition for the branes is written as a topological identity of non-commutative gauge theory. This identifies stable B-branes with previously proposed non-commutative instanton equations. Consistency of the non-commutative description with complex geometry is examined, using the non-linearities of the Seiberg-Witten map.

Keywords: Solitons Monopoles and Instantons, Duality in Gauge Field Theories, D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Branes of various dimensions may be regarded as submanifolds of a target space, since their position is defined by a set of boundary conditions for open strings. As open strings carry gauge fields, taking those into account promotes branes to bundles on the submanifolds. Background fields from closed strings influence the geometry of these bundles; in particular, non-commutative gauge theory becomes a valid tool for the field theory along the branes, when a B-field is turned on. Many couplings between open and closed string modes follow from the relevant star-products [1]–[5]. Branes also allow for formulations of important field-theoretic issues, such as the problem of instantons. Investigating supersymmetric Dp-branes through an effective action, Mariño, Minasian, Moore and Strominger [6] identified a class of instanton equations, and proposed a non-commutative limit thereof,

$$\hat{F}^{(0,2)} = 0\,, \tag{1.1}$$

$$\hat{F} \wedge \omega^{\frac{p-1}{2}} = 0\,. \tag{1.2}$$

On the other hand, Kapustin [7] recently used the tools of (generalized) complex geometry [8, 9, 10] to identify the influence of the B-field on topological branes. He argued that it endows B-branes on the non-commutative torus with a structure of holomorphic line bundles, the complex structure of which is dictated by the B-field and an allowed discrepancy between left and right-moving complex structures. Without using a non-commutative description, Kapustin and Li also obtained [11] a stability condition, in agreement with [6], using the world-sheet viewpoint, which makes branes emerge as boundary conditions that are compatible with a supersymmetry algebra.

Since it has been shown in string theory [12, 13] that non-commutativity predicts sensible corrections to effective actions through the Seiberg-Witten map, we are invited to put this map together with complex geometry. We may indeed ask whether (1.1) and (1.2) can be recovered by taking the Seiberg–Witten map of current-matching conditions.

The structure of the present paper is as follows. In section 2, I review the geometric set-up that is subject to boundary conditions. In the presence of a constant B-field, this yields a well-defined candidate for the complex structure in terms of which (1.1) can be written. In section 3, the matching of spectral flows that amounts to the requirement of stability is written in terms of a constant phase. Supposing a consistent non-commutative set-up exists, I propose to lift this quantity to non-commutative gauge theory. This leads to (1.2) in the Seiberg-Witten limit, when certain non-linearities are neglected (these restrictions will be discussed in the investigation of the Seiberg–Witten map). Section 4 finally addresses the question of whether (1.1) holds for a non-constant field strength, thus investigating the existence of the non-commutative holomorphic line bundle.

2. Generalized complex geometry and boundary conditions for B-branes

Consider a B-brane as a line bundle over a complex torus X, of real dimension p. Two geometric facts will be of central importance in the discussions. First, left and right-moving complex structures I_+ and I_- may be regarded as data of the superconformal theory. Left and right-moving Kaehler forms ω_+ and ω_-, and currents are associated to them. Second, these currents are subject to boundary conditions, which embody the presence of a brane. This will provide us with equations relating left and right-moving data. Boundary conditions induce a natural candidate for the complex structure emerges, which amounts to a sense of holomorphicity for line bundles.

Boundary conditions for fermions, once written in terms of the linear combinations $\psi^i_+ + \psi^i_-$ and $\psi^i_+ - \psi^i_-$, lead naturally to the definition of

$$\begin{aligned}\psi^i &:= \frac{1}{2}(\psi^i_+ - \psi^i_-)\,,\\ \rho_i &:= \frac{1}{2}G_{ij}(\psi^i_+ + \psi^i_-)\,,\end{aligned}$$

sections of the tangent and cotangent bundle of the brane respectively. They take the form

$$\rho_i = -(B+F)_{ij}\psi^j\,.$$

Considering the block-diagonal complex structure suited to ψ_+ and ψ_-

$$\begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix}$$

and performing the change of variables described above in the case of vanishing field strength, we obtain a non-diagonal complex structure $\mathcal{I}$ on $TX \oplus TX^*$

$$\mathcal{I} = \begin{pmatrix} \tilde{I} & -\delta P \\ \delta\omega & -\tilde{I}^t \end{pmatrix},$$

JHEP10(2004)002

where the mean complex structure and the discrepancies between the various tensors are denoted as

$$\begin{aligned}\tilde{I} &:= \frac{1}{2}\left(I_+ + I_-\right) ,\\ \delta\omega &:= \frac{1}{2}\left(\omega_+ - \omega_-\right) ,\\ \delta P &:= \frac{1}{2}\left(\omega_+^{-1} - \omega_-^{-1}\right) .\end{aligned}$$

Consider the eigenproblem

$$\mathcal{I}(\rho,\psi) = i(\rho,\psi) .$$

Imposing the boundary condition $\rho = -B\psi$ yields two equations in ψ only, which by inner consistency yields a constraint on the discrepancies between left and right-moving objects:

$$\delta\omega + B(\delta P)B + B\tilde{I} + \tilde{I}^t B = 0 . \tag{2.1}$$

Once the constraint is satisfied, the equation in ψ may be read as an eigenproblem defining the $(0,1)$ part of the tangent space, in terms of the complex structure

$$I := \tilde{I} + (\delta P)B .$$

Let us write the matching condition of left and right-moving R-currents in the above notations:

$$\omega_{+ij}\psi_+^i\psi_+^j = \omega_{-ij}\psi_-^i\psi_-^j$$

i.e.

$$\delta\omega(\psi,\psi) + \delta P(\rho,\rho) + 2(\rho,\tilde{I}\psi) = 0 ,$$

which under $\rho = -B\psi$ reduces to the condition on discrepancy between Kaehler forms (2.1).

Of course, wherever the gauge field is fluctuating (where a non-trivial curvature F is present), the boundary conditions have to be rewritten in terms of the gauge-invariant combination $B+F$ substituted to B. Then of course, the line bundle is not holomorphic with respect to I in the commutative set-up, and instead one obtains a quadratic failure:

$$FI + I^t F = -F\delta P F . \tag{2.2}$$

This failure was conjectured in [7] to be cured by the Seiberg–Witten map, which should induce the following constraint on the non-commutative field strength:

$$\hat{F}^{(0,2)} = 0 .$$

This proposal was justified for the case of constant field strength. The further investigation of spatially varying field strength is interesting for at least two reasons: non-trivial gauge field configurations include varying field strengths (we will in fact assume, as in [6], that F is localized on some region, and use the fact that $B+F$ reduces to B away from this region), furthermore the non-local nature of non-commutative field theory makes it sensitive to the variations of fields through the properties of star-products.

JHEP10(2004)002

3. Lifting the stability condition to non-commutative gauge theory

Let us first review the mapping between the commutative and non-commutative descriptions of the Ramond–Ramond couplings to Dp-branes in flat space. This will prepare us to recognize the stability condition as a topological identity, and give us a recipe for its non-commutative version.

The Seiberg-Witten map is expressed as follows in the case of constant curvature:

$$F = \hat{F}\frac{1}{1-\theta\hat{F}} .$$

The recipe for getting the full map in the case of varying field strength is to consider each of the Fourier modes separately. To the mode k_μ is associated a vector $\theta^{\mu\nu}k_\nu$ reflecting the non-locality of non-commutative field theory and the possible failure of gauge invariance. Gauge invariance is then restored [14, 15, 16, 17, 18, 19] by path-ordering with respect to the Moyal star-product $*$ along a straight open Wilson line W_k, whose extension is precisely given by the vector $\theta^{\mu\nu}k_\nu$. The Pfaffian factor $\sqrt{\det(1-\theta\hat{F})}$ comes from the integration measure can be motivated by studying the non-commutative Dirac–Born–Infeld action [16]:

$$\begin{aligned}
W_k(x) &:= \exp\left(i\int_0^1 \theta^{\mu\nu}k_\nu\hat{A}_\mu(x+\theta k\tau)\right) , \\
F_{\mu\nu}(k) &= \int dx \int d\tau_1 \times \\
&\quad \times \int d\tau_2 P_*\left(\sqrt{\det(1-\theta\hat{F})}(x+\theta k\tau_1)\left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)_{\mu\nu}(x+\theta k\tau_2)W_k(x)\right) * e^{ikx} \\
&= \int dx L_*\left(\sqrt{\det(1-\theta\hat{F})}, \left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)_{\mu\nu}, W_k(x)\right) * e^{ikx} ,
\end{aligned}$$

where P_* denotes path-ordering with respect to the Moyal product, and L_* denotes the smearing prescription along the Wilson line W_k.

The Seiberg-Witten map ensures agreement between commutative and non-commutative descriptions of couplings to C^{p-1}. When writing down couplings to Ramond-Ramond fields of lower degree,[1] more field strengths are inserted along the Wilson line:

$$\sum_n C^{(n)}(-k)\wedge\int dx\, \left(e^{B+F}\, e^{ikx}\right) = \sum_n C^{(n)}(-k)\wedge\int dx\, L_*\left(\frac{\mathrm{PfQ}}{\mathrm{Pf}\theta}, e^{Q^{-1}}, W_k(x)\right) * e^{ikx} , \tag{3.1}$$

where Q is the (inverse of the) non-commutative counter-part of the symplectic structure $B+F$, in the case of constant field strength:

$$Q^{ij} = \left(\frac{1}{B+F}\right)^{ij} = \theta^{ij} - \theta^{ik}\hat{F}_{kl}\theta^{lj} .$$

[1] Integration is along the $p+1$-dimensional world-volume, and one only retains the couplings in which the sum of form-degrees from Ramond-Ramond fields and gauge fields equals $p+1$.

JHEP10(2004)002

Furthermore, the coupling to the top-form C^{p+1} yields one more identity, although no field strength can carry form-indices in this coupling. For a flat brane, the commutative coupling is a zero-form that does not depend on the field strength. On the other hand, the non-commutative expression for this coupling still has to be gauge-invariant, and to involve additional gauge fields, even if all indices are contracted. This provides [5, 16] an identity between a gauge-invariant non-commutative expression, and a commutative expression that is actually more than gauge-invariant. The commutative side does not know about the variations of the non-commutative field strength, hence the name *topological identity.* It comes out as the zero-form part of the identity (3.1) between Ramond–Ramond couplings:

$$\delta(k) = \int dx\ L_* \left(\sqrt{\det\left(1 - \theta \hat{F}\right)}, W_k(x) \right) * e^{ikx} .$$

This situation, where a gauge-invariant quantity is actually a constant, is reminiscent of the presence of a constant factor in the requirement of stability. The biais introduced by the complex structure forces us to adapt the above procedure to the case at hand, endowed with complex geometry. We are allowed to repeat it in our context through the following steps:

1. consider a gauge-invariant quantity depending on commutative gauge fields and B-fields,

2. write it in terms of non-commutative variables in the case of constant field strength,

3. restore gauge invariance by the smearing prescription.

Let us consider the stability condition for B-branes and look for a non-commutative version thereof, postponing the discussion of the very existence of the non-commutative holomorphic set-up (encoded in (1.1) to the next section 4.

The stability condition involves a proportionality factor $e^{i\alpha}$ describing the combination of supersymmetries that is preserved in presence of the brane. In the world-sheet approach, spectral flows are required by stablity to match, up to this phase. This fact is expressed in terms of the holomorphic n-form Ω as

$$\Omega_{i_1 \ldots i_n} \psi_+^{i_1} \cdots \psi_+^{i_n} = e^{i\alpha}\, \Omega_{i_1 \ldots i_n} \psi_-^{i_1} \cdots \psi_-^{i_n} .$$

Let us supplement it with the boundary condition

$$\psi_+ = R \psi_- ,$$

where along the tangent space of the brane, $R = (g - F)^{-1}(g + F)$. Once restricted to the holomorphic sector with respect to I, the matching condition [11] reduces to

$$\frac{\det(\omega + iF)}{\det(g + F)} = e^{i\alpha} ,$$

where $\omega = gI$. This is the world-sheet analog of the space-time derivation of [6], and agrees with its result. But here we recognize the situation that has been used in non-commutative field theory to produce a topological identity. The LHS is indeed a gauge-invariant quantity, while the RHS is actually more than gauge-invariant. As a constant, it is topological in nature and should not depend on any change of variables on the LHS. In particular, provided the Seiberg–Witten map is consistent with holomorphicity, the non-commutative counter-part of the LHS will still equal the phase $e^{i\alpha}$. This legitimates the approach in [6], where the non-commutative counter-part was postulated, and the constant evaluated by considering a region where the curvature is zero. In the Seiberg–Witten limit this yields

$$\frac{(B+F)^{\frac{p-2}{2}} \wedge \omega}{\mathrm{Pf}(B+F)} = \frac{B^{\frac{p-2}{2}} \wedge \omega}{\mathrm{Pf} B},$$

which together with the Seiberg-Witten map completes the first two steps described above to give

$$\hat{F} \wedge \omega^{\frac{p-2}{2}} = 0 .$$

However, the last step involves a smearing prescription that would dictate higher non-linearities in the gauge field. The non-commutative proposal of [6] is therefore to be regarded as a version of lowest possible order in the gauge field, and all orders in derivatives. This way of treating the Wilson line as a mere smearing prescription, disregarding additional non-linearities from $\hat{A}$, will be paralleled in our discussion of holomorphicity in the next section 4, and restrict explicit computations to low order in the non-commutative gauge field.

So far we have obtained a natural lift of the stability condition to non-commutative gauge theory, but this result has to be supplemented with a test of the very existence of non-commutative gauge theory, in a compatible way with the holomorphicity used above. Indeed, we have to show that the Seiberg-Witten map can be performed so as to yield a holomorphic line bundle on the non-commutative torus. This involves taking the Seiberg-Witten map of (2.2) to prove that

$$\hat{F}^{(0,2)} = 0 ,$$

which will complete the world-sheet derivation of the non-commutative equations (1.1) and (1.2).

4. Compatibility with the Seiberg-Witten map

In the presence of a non-flat gauge bundle, the boundary conditions contain a term linear in F, that leads to the quadratic failure (2.2) of the gauge bundle to be holomorphic with respect to I in the commutative set-up. Due to the non-linearity of the mapping from commutative to non-commutative gauge field, it is very tempting to hope for a cancellation between the quadratic failure and corresponding non-linearities in the Seiberg–Witten map. Star-products inherited from the Moyal products induce an infinite number of derivatives in all the expressions, making them suited to non-constant fields.

JHEP10(2004)002

Our strategy will therefore be the following: start with (2.2) in the commutative set-up, map it to non-commutative gauge theory, expand it in powers of $\hat{A}$, keeping all orders in derivatives, and ask whether holomorphicity can ensure consistency at fixed order in $\hat{A}$. Various n-ary $*_n$-products, introduced at low orders by Mehen and Wise [20] and at all orders by Liu [14], are useful whenever a mapping to non-commutative gauge theory is expanded in powers of the gauge field. Their definition is tightly related to gauge invariance. We have already written down the couplings to Ramond–Ramond form fields. More generally, given a set of local operators O_i, the following quantity is gauge-invariant

$$Q(k) = \int dx\, L_* \left(O_1, \ldots, O_n, W_k(x)\right) * e^{ikx} ,$$

and may be expanded in powers of the non-commutative gauge field $\hat{A}$, thereby defining the $*_n$-products:[2]

$$Q(k) = \sum_{m \geq 0} Q_m(k) ,$$

where

$$Q_m(k) = \frac{1}{m\,!} (\theta\partial)^{\mu_1} \ldots (\theta\partial)^{\mu_m} \langle O_1, \ldots, O_n, \hat{A}_{\mu_1}, \ldots, \hat{A}_{\mu_m} \rangle_{*_{n+m}} .$$

Each of the products contains derivatives of arbitrarily high order an so that not surprisingly an infinite number of derivatives show up even at low orders in the gauge fields.

The full Seiberg-Witten map therefore contains two types of additional complications. The first one, that will be treated in detail, comes the infinity of derivatives that acts on fields through the star-products. The second one comes from the contribution of the gauge fields from the Wilson line, that can be of arbitrarily high degree and whose contribution will be sketched. In other words, the effect of the Wilson line will be approximated by the smearing prescription. As announced above, this actually corresponds to the degree of precision of the non-commutative Hermitian Yang-Mills equations as proposed in [6], where the Wilson line does not appear. A full proof of $\hat{F}^{(0,2)} = 0$ should be paired with a smeared version of (1.2),

$$\int dx\, L_* \left(\sqrt{\det(1 - \theta \hat{F})}, \hat{F} \wedge \omega^{\frac{p-1}{2}}, W_k(x) \right) * e^{ikx} = 0 . \tag{4.1}$$

In order to obtain constraints on the non-commutative gauge theory from the constraint (2.2) on the field strength, we must formulate this condition in terms of non-commutative fields in position space. The rank-one Seiberg-Witten map in flat space [20, 14, 15] admits a natural formulation in momentum space, since it involves a straight open Wilson line W_k whose extension $\theta^{\mu\nu} k_\nu$ depends on the Fourier mode k_μ in consideration:

$$F_{ij}(k) = \int dx\, L_* \left(\sqrt{\det(1 - \theta \hat{F})},\, \hat{F}_{il} \left(\frac{1}{1 - \theta \hat{F}} \right)^l_j ,\, W_k(x) \right) * e^{ikx} , \tag{4.2}$$

[2]Expressions for the differential operators can be found in [14], and any order n can be recursively obtained by requiring gauge invariance

where L_* denotes the smearing prescription: it averages over all the possible ways of inserting operators along the Wilson line. The LHS of the equation

$$FI + I^t F = -F\delta PF$$

only contains one operator in position space. We may thus rewrite it, using the Seiberg-Witten map, as the Fourier transform of the expression

$$\int dx\, L_*\left(\sqrt{\det(1-\theta\hat{F})}, \left(\hat{F}\frac{1}{1-\theta\hat{F}}I + I^t\hat{F}\frac{1}{1-\theta\hat{F}}\right), W_k(x)\right) * e^{ikx}\,, \qquad (4.3)$$

because objects carrying no index are transparent to the endomorphism I. The RHS is more involved, because it consists of a pointwise product of two fields of gauge theory in position space, namely F and δPF. Its Fourier transform at momentum k_μ will therefore be a convolution of two Wilson lines, whose extensions sum to $\theta^{\mu\nu}k_\nu$, each of these Wilson lines having one of the fields $\hat{F}(1-\theta\hat{F})^{-1}$ and $\hat{F}(1-\theta\hat{F})^{-1}$ attached at its beginning. This is equivalent to the concatenation of two Wilson lines into one, of extension $\theta^{\mu\nu}k_\nu$, all the observables being smeared along the result. For the mode k_μ, we obtain in momentum space

$$\int dx\, L_*\left(\sqrt{\det(1-\theta\hat{F})}, \hat{F}\frac{1}{1-\theta\hat{F}}, \left(\delta P\hat{F}\frac{1}{1-\theta\hat{F}}\right), W_k(x)\right) * e^{ikx}\,. \qquad (4.4)$$

Since differential operators $*_n$ enable us to expand both expressions (4.3) and (4.4) at all orders in derivatives and any finite order in the gauge field, we will read off the conditions for the expansions of the two sides of (2.2) to match order by order in the gauge field, even for varying fields. To this end, we are instructed to plug the following quadratic expansion of the Seiberg–Witten map (4.2) into the equality condition between (4.3) and (4.4):

$$F_{ij} = \hat{F}_{ij} + \theta^{mn}\langle\hat{F}_{im},\hat{F}_{nj}\rangle_{*_2} - \frac{1}{2}\theta^{mn}\langle\hat{F}_{nm},\hat{F}_{ij}\rangle_{*_2} + \theta^{mn}\partial_n\langle\hat{A}_m,\hat{F}_{ij}\rangle_{*_2} + O(\hat{F}^3)\,. \qquad (4.5)$$

The first order in the field strength is equivalent to (1.1) since (4.4) does not contain any linear term in the curvature:

$$\hat{F}I + I^t\hat{F} = o(\hat{F})\,.$$

Does the holomorphicity condition still ensure consistency at quadratic order, where the failure enters the equation? The quadratic order namely makes a single term appear on the RHS, with each of the curvatures carrying one index:

$$\hat{F}_{il}(I\theta + \theta I^t)^{lm}\hat{F}_{mj}$$

because each of the form indices has to be brought by a field strength, with no contribution either from the determinant or from the denominators. It has a counter-part[3] on the LHS, through the term $\theta^{mn}\langle\hat{F}_{im},\hat{F}_{nj}\rangle_{*_2}$ in the expansion of the Seiberg–Witten map in position space.

[3]This step requires the equation $\delta P = I\theta + \theta I^t$ of [7], in order to have the same tensors contracted with field on both sides.

JHEP10(2004)002

Now we can read off the condition to be fulfilled in order for the two sides to match at quadratic order in the gauge field. It is given by the vanishing of those quadratic terms induced by the expansion (4.5) in the LHS, that have not been cancelled by the quadratic expansion of $F\delta PF$:

$$0 = -\frac{1}{2}\theta^{mn}\langle \hat{F}_{nm}, (\hat{F}I + I^t\hat{F})_{ij}\rangle_{*_2} + \theta^{mn}\partial_n\langle \hat{A}_m, (\hat{F}I + I^t\hat{F})_{ij}\rangle_{*_2}\,.$$

We observe that it is identically verified if the holomorphicity condition obtained at the linear step is fulfilled. This is the first test of the non-commutative holomorphicity proposal passed by varying fields.

We see that expanding at quadratic order in the gauge field involves contributions from the Pfaffian and from the Wilson line on the LHS, whereas the gauge fields on the RHS only come from the observables $\hat{F}(1-\theta\hat{F})^{-1}$ and $\delta P\hat{F}(1-\theta\hat{F})^{-1}$. The scheme for higher-order terms in the gauge fields to cancel each other is therefore of the following recursive form: suppose the holomorphicity condition has been shown to ensure the consistency of the Seiberg–Witten map of (2.2) up to p-th order in the gauge field (we have just done this for $p = 2$). Then, going from order p to order $p + 1$ will involve one more curvature on the RHS, cancelling only part of the new terms generated on the LHS. Those new terms on the LHS containing one more potential $\hat{A}$ will find no counter-part on the RHS and will only disappear if the holomorphicity condition holds. Of course, as soon as $p \geq 3$, gauge potentials from the Wilson line on the RHS will appear, corresponding to the smeared version (4.1) of the stability condition (1.2).

5. Conclusions

We have put together a proposed non-commutative version of the instanton equations obtained from supersymmetric D-branes using effective actions, and the stable holomorphic line bundles that emerge from the study of topological branes on the non-commutative torus. We thereby established more firmly the relevance of non-commutative gauge theory for the study of instantons. On the way we identifed one of the equations as a topological identity of non-commutative gauge theory, associated to the existence of a constant factor in the stability condition. Complex structures fit into techniques involved in generating topological indentities and derivative corrections to effective actions from non-commutative field theory. This was to be awaited since an alternative world-sheet derivation should exist for the results based on the study of branes as supersymmetric solitons.

Nevertheless, the full deformation derived in [6] are a two-parameter deformation of usual instanton equations. The limit we investigated corresponds to the Seiberg-Witten limit of non-commutative gauge theory. It is a very peculiar case, just as the derivative corrections to effective actions derived from non-commutativity in the Seiberg–Witten limit are a subset of those that can be derived by going beyond this limit, further deforming the star-product. In a sense, dealing with the non-linearities as we did amounts to taking into account a more precise effective action, involving $*_n$-products between fields entering

JHEP10(2004)002

couplings of rank $2n$. Recovering the two-parameter deformation should involve the deformations $\tilde{*}_n$ that appeared in [21, 22]. Very roughly, we can note that the first contribution from the metric to these products is quadratic, just as in the deformed instanton equations of [6].

The issue of the inclusion of scalars in stability conditions has been investigated [23], but exhibiting a non-commutative version would require the knowledge of an explicit solution to the Seiberg–Witten equations of non-Abelian gauge theory. On the other hand, describing an Abelian D-brane in terms of lower-dimensional ones in the language of matrix theory allows for identities involving scalars, since the Abelian D-brane does not couple à la Myers [24] to transverse scalars. It should be possible to obtain from a topological identity an instanton equation including scalars, that would show up as infinite matrices.

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